HOLLOW SPHERE OF RANDOMLY BONDED MATERIAL SUBJECTED TO INTERNAL PRESSURE

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A hollow sphere of an elastic binding material of slight stiffness, bonded randomly by "fiber" segments of a stiffer material, is considered. Polymer material, for example, can be the binder. Such a bonding permits obtaining a material with improved properties, where the material on the whole is quasi-isotropic [1]. The stress distribution in a hollow sphere is obtained.

Let a composite medium consist of an elastic binder and an armature in the form of segments of circular cylindrical fibers. Let us assume the fiber diameter d to be considerably less than their length l ($d \ll l$). Following [2], let us introduce the following as initial hypotheses.

1. Let the fiber segments be distributed uniformly in all directions in the binder material. Let us identify the macrovolume ω ($\omega \ll V$; V is the body volume) with a material point. The number of fiber segments in the volume ω under consideration is sufficiently large. A uniaxial stress state is realized in each fiber. Let us consider the bonded material as a macroscopically homogeneous medium. Let σ_{ij}, ε_{ij} (i, j = 1, 2, 3) denote, respectively, the stress and strain tensor components in a rectangular x_1, x_2, x_3 coordinate system.

2. Let us assume that the binder material is deformed elastically. Let λ_c, μ_c denote the Lamé constants of the binder.

3. The stess–strain dependence in the armature is nonlinear and is given by the equation $\sigma_{nm} = F(\varepsilon_{nm})$, where ε_{nm} is the axial strain of the fibers and σ_{nm} is the axial stress.

Let Ω be a hemisphere formed by the unit vectors **n**, directed along the fiber axis. The relative volume of the fibers for which the vector **n** is within the solid angle $d\Omega$ is proportional to $d\Omega$ and 2π-fold less than the volume of all the fibers. Let n_1, n_2, n_3 denote the direction cosines of the vector **n** in the x_1, x_2, x_3 coordinate system and η the coefficient of volume content of the armature in the material. Assuming the strains homogeneous and taking the hypothesis about the volume contribution of the components to the total stress state, we obtain the following relationship between the stress and strain:

$$\sigma_{ij} = (1-\eta)(\lambda_c \varepsilon \delta_{ij} + 2\mu_c \varepsilon_{ij}) + \frac{\eta}{2\pi} \iint_{\Omega} F(\varepsilon_{nn}) n_i n_j d\Omega. \quad (1)$$

Here $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$; $\varepsilon_{nm} = \varepsilon_{ij} n_i n_j$ (i, j = 1, 2, 3).

Let us examine the case when the function F is

$$F(\varepsilon_{nn}) = \begin{cases} E\varepsilon_{nn} & \text{for} \quad \varepsilon_c < \varepsilon_{nn} < \varepsilon_t; \\ 0 & \text{for} \quad \varepsilon_{nn} \leqslant \varepsilon_c \quad \text{or} \quad \varepsilon_t \leqslant \varepsilon_{nn}, \end{cases} \quad (2)$$

i.e., the fibers are deformed according to Hooke's law under tension to a strain ε_t and under compression to a strain ε_c. Reaching these limit strains results in brittle fracture of the fibers. Here E is Young's modulus of the armature.

Let us assume that the armature works elastically, i.e., fiber fracture has not yet occurred. Taking account of the dependence $F(\varepsilon_{nm}) = E\varepsilon_{nm}$, we obtain the elastic relation (Hooke's law) between the stress

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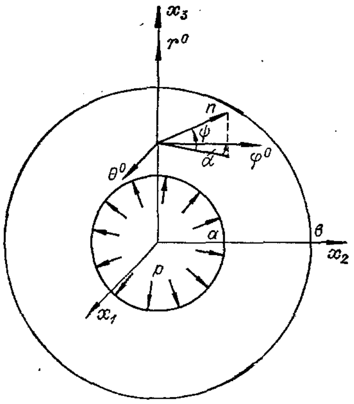

Fig. 1

and strain tensors from (1). The Lamé coefficients of the composite material λ_k and μ_k will equal in this case

$$\lambda_k=(1-\eta)\ \lambda_c+1/15\eta E;$$
$$\mu_k=(1-\eta)\mu_c+1/15\eta E.$$

It is more convenient to solve the problem in the r, θ, φ spherical coordinate system. Because of symmetry, it is sufficient to consider the problem for one fixed ray directed along the sphere radius, along the x_3 axis, say. Let us direct the unit vectors $\mathbf{r}^0$, $\boldsymbol{\theta}^0$, $\boldsymbol{\varphi}^0$ of the spherical coordinate system parallel to the $0x_3$, $0x_1$, $0x_2$ axes respectively. Let us rewrite (1) in this coordinate system by introducing the angles α and ψ (Fig. 1), which give the direction of the vector **n**. Let us hence omit the subscripts on the λ_c and μ_c:

$$\sigma_r=(1-\eta)(\lambda\varepsilon+2\mu\varepsilon_r)+\eta\int_0^{\pi/2}F(\varepsilon_{nn})\sin^2\psi\cos\psi d\psi; \tag{3}$$
$$\sigma_\theta=(1-\eta)(\lambda\varepsilon+2\mu\varepsilon_\theta)+\frac{\eta}{2\pi}\iint_\Omega F(\varepsilon_{nn})\sin^2\alpha\cos^2\psi d\Omega;$$
$$\sigma_\varphi=(1-\eta)(\lambda\varepsilon+2\mu\varepsilon_\varphi)+\frac{\eta}{2\pi}\iint_\Omega F(\varepsilon_{nn})\cos^2\alpha\cos^2\psi d\Omega.$$

Because of the symmetry of the problem, $\varepsilon_\theta=\varepsilon_\varphi$, $\sigma_\theta=\sigma_\varphi$ and the components with different subscripts are zero [3]. The axial strain of the fibers is

$$\varepsilon_{nn}=\varepsilon_r\sin^2\psi+\varepsilon_\theta\cos^2\psi\sin^2\alpha+\varepsilon_\varphi\cos^2\psi\cos^2\alpha.$$

Since $\varepsilon_\theta=\varepsilon_\varphi$, then it is possible to write

$$\varepsilon_{nn}=\varepsilon_r\sin^2\psi+\varepsilon_\theta\cos^2\psi. \tag{4}$$

Therefore, ε_{nm} is independent of the angle α. The dependence on α also vanishes for the sum $\sigma_\theta+\sigma_\varphi$

$$\sigma_\theta+\sigma_\varphi=2(1-\eta)[\lambda\varepsilon+\mu(\varepsilon_\theta+\varepsilon_\varphi)]+\eta\int_0^{\pi/2}F(\varepsilon_{nn})\cos^3\psi d\psi. \tag{5}$$

Taking account of (2), let us rewrite (3), (5) as follows:

$$\sigma_r=(1-\eta)(\lambda\varepsilon+2\mu\varepsilon_r)+\eta\int_\gamma^\delta E\varepsilon_{nn}\sin^2\psi\cos\psi d\psi; \tag{6}$$
$$\sigma_\theta+\sigma_\varphi=2(1-\eta)[\lambda\varepsilon+\mu(\varepsilon_\theta+\varepsilon_\varphi)]+\eta\int_\gamma^\delta E\varepsilon_{nn}\cos^3\psi d\psi.$$

The angles γ and δ are determined from the relationships

$$\varepsilon_\gamma\equiv\varepsilon_r\sin^2\gamma+\varepsilon_\theta\cos^2\gamma=\varepsilon_t;$$
$$\varepsilon_\delta\equiv\varepsilon_r\sin^2\delta+\varepsilon_\theta\cos^2\delta=\varepsilon_c.$$

We hence obtain

$$\sin^2\gamma=(\varepsilon_t-\varepsilon_\theta)/(\varepsilon_r-\varepsilon_\theta);$$
$$\sin^2\delta=(\varepsilon_\theta-\varepsilon_c)/(\varepsilon_\theta-\varepsilon_r).$$

Therefore

$$\gamma=\begin{cases}0 \text{ for } & \varepsilon_\gamma<\varepsilon_p;\\ \arcsin\left[(\varepsilon_t-\varepsilon_\theta)/(\varepsilon_r-\varepsilon_\theta)\right]^{1/2} \text{ for } & \varepsilon_\gamma=\varepsilon_t;\end{cases}$$
$$\delta=\begin{cases}\pi/2 \text{ for } & \varepsilon_\delta>\varepsilon_c;\\ \arcsin\left[(\varepsilon_\theta-\varepsilon_c)/(\varepsilon_\theta-\varepsilon_r)\right]^{1/2} \text{ for } & \varepsilon_\delta=\varepsilon_c.\end{cases}$$

Integrating (6) with respect to ψ and taking account of (4), we obtain

$$\sigma_r=(w+g)\varepsilon_r+(v+c)\varepsilon_\theta; \tag{7}$$
$$\sigma_\theta+\sigma_\varphi=(v+c)\varepsilon_r+(2w+v+t)\varepsilon_\theta,$$

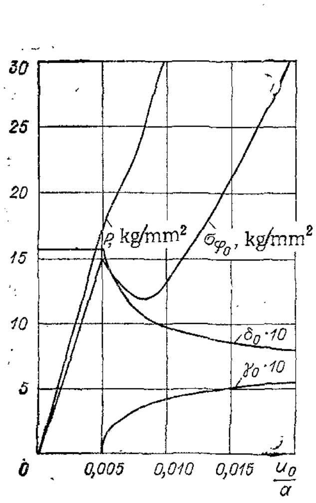

Fig. 2

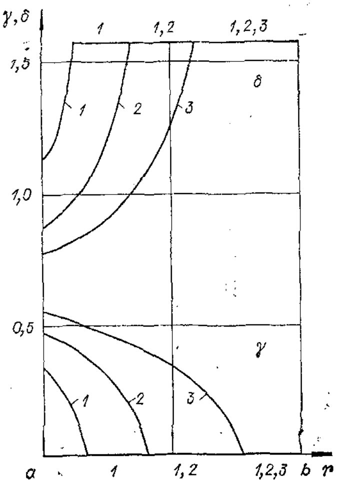

Fig. 3

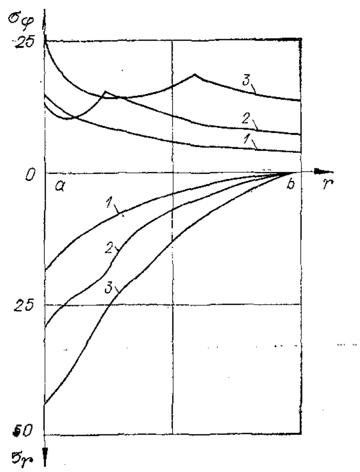

Fig. 4

where

$$v=2(1-\eta)\lambda;\quad w=(1-\eta)(\lambda+2\mu);$$
$$g=1/5\eta E(\sin^2\delta-\sin^2\gamma);$$
$$c=1/5\eta E[\sin^3\delta(\cos^2\delta+2/3)-\sin^3\gamma(\cos^2\gamma+2/3)];$$
$$t=1/5\eta E[25/8(\sin\delta-\sin\gamma)+25/48(\sin 3\delta-\sin 3\gamma)+1/16(\sin 5\delta-\sin 5\gamma)].$$

Let us express the strain in terms of the displacement u [3]:

$$\varepsilon_r=\frac{du}{dr};\quad \varepsilon_\theta=\varepsilon_\varphi=u/r$$

and let us substitute into (7):

$$\sigma_r=(w+g)\frac{du}{dr}+(v+c)\frac{u}{r};\qquad (8)$$
$$\sigma_\theta+\sigma_\varphi=(v+c)\frac{du}{dr}+(2w+v+t)\frac{u}{r}.$$

Since λ and μ are positive and δ is greater than γ, then $w+g>0$. The equilibrium equation in a spherical coordinate system is [3]

$$r\frac{d\sigma_r}{dr}+2(\sigma_r-\sigma_\theta)=0.\qquad (9)$$

However, it is more convenient to convert it into

$$\frac{d\sigma_r}{dr}+\frac{2\sigma_r-(\sigma_\theta+\sigma_\varphi)}{r}=0.\qquad (10)$$

Since $\sigma_\theta=\sigma_\varphi$, then (9) always follows from (10), but the angle α does not enter into (10). Let us substitute (9) into (10). After simple manipulations, we obtain

$$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}-\frac{2}{r^2}u+\frac{1}{w+g}\left[\frac{dg}{dr}\frac{du}{dr}+\left(\frac{1}{r}\frac{dc}{dr}+\frac{2g+c-t}{r^2}\right)u\right]=0.\qquad (11)$$

Let us take the boundary conditions in the form

$$r=a:\quad u(a)=U;\qquad (12)$$
$$r=b:\quad \sigma_r(b)=0.$$

Let us solve the problem (11), (12) numerically by introducing the iteration in u since (11) is nonlinear. Let us approximate (11) by a difference scheme to second-order accuracy [4]:

$$\frac{u_{k+1}^{(i+1)}-2u_k^{(i+1)}+u_{k-1}^{(i+1)}}{h^2}+\frac{2}{r_k}\frac{u_{k+1}^{(i+1)}-u_{k-1}^{(i+1)}}{2h}-\frac{2}{r_k^2}u_k^{(i+1)}+f_k^{(i)}=0,\qquad (13)$$

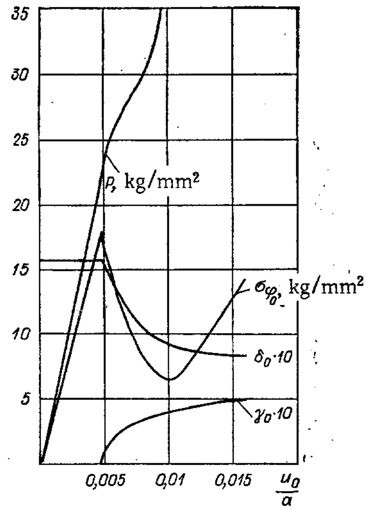

Fig. 5

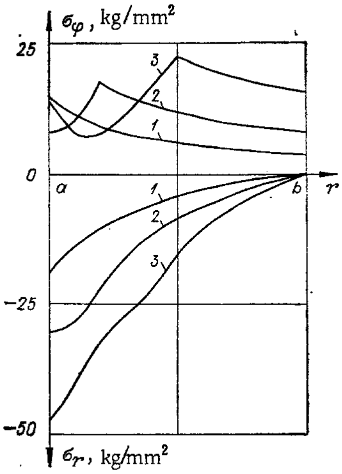

Fig. 6

where

$$f_k^{(i)} = \frac{1}{w + g_k^{(i)}} \left[\frac{g_{k+1}^{(i)} - g_{k-1}^{(i)}}{2h} \cdot \frac{u_{k+1}^{(i)} - u_{k-1}^{(i)}}{2h} + \left(\frac{1}{r_k} \frac{c_{k+1}^{(i)} - c_{k-1}^{(i)}}{2h} + \frac{2g_k^{(i)} + c_k^{(i)} - t_k^{(i)}}{r_k^2} \right) u_k^{(i)} \right].$$

The subscript denotes the number of the point on the sphere radius (k=1, 2, ..., n−1), while the superscript in the parentheses denotes the number of the iteration step at which the value of the function is taken.

Let us rewrite (13) as follows:

$$A_k u_{k-1}^{(i+1)} - C_k u_k^{(i+1)} + B_k u_{k+1}^{(i+1)} = -F_k^{(i)}.$$

Here

$$A_k = 1 - \frac{h}{r_k}; \quad B_k = 1 + \frac{h}{r_k}; \quad C_k = 2\left(1 + \frac{h^2}{r_k^2}\right); \quad F_k^{(i)} = h^2 f_k^{(i)}. \tag{14}$$

Equating σ_r in (8) to zero according to (12), we obtain

$$\left.\frac{du}{dr}\right|_{r=b} + \frac{v+c}{w+g} \frac{u(b)}{b} = 0.$$

Let us approximate the boundary conditions by a difference scheme with second-order accuracy:

$$u_0 = U;$$
$$\frac{u_{n-2}^{(i+1)} - 4u_{n-1}^{(i+1)} + 3u_n^{(i+1)}}{2h} + \varkappa u_n^{(i+1)} = 0, \tag{15}$$

where

$$\varkappa = \frac{v + C_n^{(i)}}{b\left(w + g_n^{(i)}\right)}.$$

Let us seek the solution by the method of left factorization [4]. Using the second equation from (15), we find the factorizing coefficients $\xi_n^{(i+1)}$ and $\eta_n^{(i+1)}$:

$$\xi_n^{(i+1)} = \frac{4A_{n-1} - C_{n-1}}{(3 + 2h\varkappa) A_{n-1} - B_{n-1}};$$

$$\eta_n^{(i+1)} = \frac{\left[4 - (3 + 2h\varkappa)\xi_n^{(i+1)}\right] F_{n-1}^{(i)}}{(3 + 2h\varkappa) C_{n-1} - 4B_{n-1}}.$$

Here A_{n-1}, B_{n-1}, C_{n-1}, and $F_{n-1}^{(i)}$ are expressed by means of (14).

We determine u_0 from the first equation in (12). We then find the remaining displacements $u_k^{(i+1)}$ (k=1, 2, ..., n). We determine ε_{r_k}, ε_{θ_k}, ε_{φ_k} in terms of $u_k^{(i+1)}$:

$$\varepsilon_{r_0} = \frac{-3u_0 + 4u_1^{(i+1)} - u_2^{(i+1)}}{2h};$$

$$\varepsilon_{r_n} = \frac{u_{n-2}^{(i+1)} - 4u_{n-1}^{(i+1)} + 3u_n^{(i+1)}}{2h};$$

$$\varepsilon_{r_k} = \frac{u_{k+1}^{(i+1)} - u_{k-1}^{(i+1)}}{2h} \quad (k = 1, \ldots, n-1);$$

$$\varepsilon_{\theta_k} = \varepsilon_{\varphi_k} = \frac{u_k^{(i+1)}}{r_k} \quad (k = 0, 1, \ldots, n).$$

Finally, we determine σ_{r_k}, σ_{φ_k}, and σ_{θ_k} from (7). Since (11) is nonlinear, let us introduce an iteration in u: Let us determine $F_k^{(i+1)}$ in terms of the $u_k^{(i+1)}$ found and let us repeat the factorization without changing u_0. The counting process starts with the elastic solution. The problem was solved on the M-222 computer.

Presented in Figs. 2-4 are the results of computing a bonded sphere under the following initial data: $E=7000$ kg/mm²; $\lambda_c=300$ kg/mm²; $\mu_c=75$ kg/mm²; $\varepsilon_t=0.005$; $\varepsilon_c=0.01$; $\eta=0.1$.

Shown in Fig. 2 is the change in pressure $p=-\sigma_{r_0}$, stress σ_{φ_0}, and also the magnitudes of the angles γ_0, δ_0 on the inner surface of the sphere as a function of its displacement u_0.

Shown in Fig. 3 is the change in the angles γ and δ as a function of the radius r and pressure p (curves 1, 2, 3 correspond to $p=22.2$; 36.7; 55.3 kg/mm², respectively).

Presented in Fig. 4 are graphs showing the stress distribution in the wall of a spherical vessel as a function of the radius r and pressure p (curves 1, 2, 3 correspond to $u_0=0.005$; 0.008; 0.18A, respectively).

Shown in Figs. 5 and 6 is the change in the pressure $p=-\sigma_{r_0}$, the stress σ_{φ_0}, the magnitude of the angles γ_0, δ_0 (Fig. 5) for the following characteristics of the composite material: $E=7000$ kg/mm² $\lambda_c=300$ kg/mm²; $\mu_c=75$ kg/mm²; $\varepsilon_t=0.005$; $\varepsilon_c=-0.01$; $\eta=0.2$ (curves 1, 2, 3 correspond to $u_0=0.004$; 0.008; 0.015A).

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